SECTION 15.3 Second-Order Homogeneous Linear Equations

Second-Order Linear Differential Equations •
Higher-Order Linear Differential Equations • Applications

Second-Order Linear Differential Equations

In this section and the following section, we discuss methods for solving higher-order linear differential equations.

Definition of Linear Differential Equation of Order n

Let $g_1, g_2, \ldots, g_n$ and f be functions of x with a common (interval) domain. An equation of the form

$$y^{(n)} + g_1(x)y^{(n-1)} + g_2(x)y^{(n-2)} + \cdots + g_{n-1}(x)y' + g_n(x)y = f(x)$$

is called a **linear differential equation of order n**. If $f(x) = 0$, the equation is **homogeneous**; otherwise, it is **nonhomogeneous**.

NOTE Notice that this use of the term *homogeneous* differs from that in Section 5.7.

We discuss homogeneous equations in this section, and leave the nonhomogeneous case for the next section.

The functions $y_1, y_2, \ldots, y_n$ are **linearly independent** if the *only* solution of the equation

$$C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0$$

is the trivial one, $C_1 = C_2 = \cdots = C_n = 0$. Otherwise, this set of functions is **linearly dependent**.

EXAMPLE 1 Linearly Independent and Dependent Functions

a. The functions $y_1(x) = \sin x$ and $y_2 = x$ are linearly independent because the only values of C_1 and C_2 for which

$$C_1 \sin x + C_2x = 0$$

for all x are $C_1 = 0$ and $C_2 = 0$.

b. It can be shown that two functions form a linearly dependent set if and only if one is a constant multiple of the other. For example, $y_1(x) = x$ and $y_2(x) = 3x$ are linearly dependent because

$$C_1x + C_2(3x) = 0$$

has the nonzero solutions $C_1 = -3$ and $C_2 = 1$.

The following theorem points out the importance of linear independence in constructing the general solution of a second-order linear homogeneous differential equation with constant coefficients.

THEOREM 15.4 Linear Combinations of Solutions

If y_1 and y_2 are linearly independent solutions of the differential equation $y'' + ay' + by = 0$, then the general solution is

$$y = C_1 y_1 + C_2 y_2$$

where C_1 and C_2 are constants.

Proof We prove this theorem in only one direction. If y_1 and y_2 are solutions, you can obtain the following system of equations.

$$y_1''(x) + ay_1'(x) + by_1(x) = 0$$
$$y_2''(x) + ay_2'(x) + by_2(x) = 0$$

Multiplying the first equation by C_1, multiplying the second by C_2, and adding the resulting equations together produces

$$[C_1 y_1''(x) + C_2 y_2''(x)] + a[C_1 y_1'(x) + C_2 y_2'(x)] + b[C_1 y_1(x) + C_2 y_2(x)] = 0$$

which means that

$$y = C_1 y_1 + C_2 y_2$$

is a solution, as desired. The proof that all solutions are of this form is best left to a full course on differential equations.

Theorem 15.4 states that if you can find two linearly independent solutions, you can obtain the general solution by forming a **linear combination** of the two solutions.

To find two linearly independent solutions, note that the nature of the equation $y'' + ay' + by = 0$ suggests that it may have solutions of the form $y = e^{mx}$. If so, then $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Thus, by substitution, $y = e^{mx}$ is a solution if and only if

$$\begin{aligned} y'' + ay' + by &= 0 \\ m^2e^{mx} + ame^{mx} + be^{mx} &= 0 \\ e^{mx}(m^2 + am + b) &= 0. \end{aligned}$$

Because e^{mx} is never 0, $y = e^{mx}$ is a solution if and only if

$$m^2 + am + b = 0. \qquad \text{Characteristic equation}$$

This is the **characteristic equation** of the differential equation

$$y'' + ay' + by = 0.$$

Note that the characteristic equation can be determined from its differential equation simply by replacing y'' with m^2, y' with m, and y with 1.

EXPLORATION

For each differential equation below find the characteristic equation. Solve the characteristic equation for m, and use the values of m to find a general solution to the differential equation. Using your results, develop a general solution to differential equations with characteristic equations that have distinct real roots.

(a) $y'' - 9y = 0$

(b) $y'' - 6y' + 8y = 0$

EXAMPLE 2 Characteristic Equation with Distinct Real Roots

Solve the differential equation

$$y'' - 4y = 0.$$

Solution In this case, the characteristic equation is

$$m^2 - 4 = 0 \qquad \text{Characteristic equation}$$

so $m = \pm 2$. Thus, $y_1 = e^{m_1 x} = e^{2x}$ and $y_2 = e^{m_2 x} = e^{-2x}$ are particular solutions of the given differential equation. Furthermore, because these two solutions are linearly independent, you can apply Theorem 15.4 to conclude that the general solution is

$$y = C_1 e^{2x} + C_2 e^{-2x}. \qquad \text{General solution}$$

The characteristic equation in Example 2 has two distinct real roots. From algebra, you know that this is only one of *three* possibilities for quadratic equations. In general, the quadratic equation $m^2 + am + b = 0$ has roots

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \text{and} \qquad m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

which fall into one of three cases.

1. Two distinct real roots, $m_1 \neq m_2$
2. Two equal real roots, $m_1 = m_2$
3. Two complex conjugate roots, $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$

In terms of the differential equation $y'' + ay' + by = 0$, these three cases correspond to three different types of general solutions.

THEOREM 15.5 Solutions of $y'' + ay' + by = 0$

The solutions of

$$y'' + ay' + by = 0$$

fall into one of the following three cases, depending on the solutions of the characteristic equation, $m^2 + am + b = 0$.

1. *Distinct Real Roots* If $m_1 \neq m_2$ are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

2. *Equal Real Roots* If $m_1 = m_2$ are equal real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x} = (C_1 + C_2 x) e^{m_1 x}.$$

3. *Complex Roots* If $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$ are complex roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x.$$

EXAMPLE 3 Characteristic Equation with Complex Roots

Find the general solution of the differential equation

$$y'' + 6y' + 12y = 0.$$

Solution The characteristic equation

$$m^2 + 6m + 12 = 0$$

has two complex roots, as follows.

$$\begin{aligned} m &= \frac{-6 \pm \sqrt{36 - 48}}{2} \\ &= \frac{-6 \pm \sqrt{-12}}{2} \\ &= -3 \pm \sqrt{-3} \\ &= -3 \pm \sqrt{3}i \end{aligned}$$

Thus, $\alpha = -3$ and $\beta = \sqrt{3}$, and the general solution is

$$y = C_1 e^{-3x} \cos \sqrt{3}x + C_2 e^{-3x} \sin \sqrt{3}x.$$

NOTE In Example 3, note that although the characteristic equation has two *complex* roots, the solution of the differential equation is *real*.

EXAMPLE 4 Characteristic Equation with Repeated Roots

Solve the differential equation

$$y'' + 4y' + 4y = 0$$

subject to the initial conditions $y(0) = 2$ and $y'(0) = 1$.

Solution The characteristic equation

$$m^2 + 4m + 4 = (m + 2)^2 = 0$$

has two equal roots given by $m = -2$. Thus, the general solution is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}. \qquad \text{General solution}$$

Now, because $y = 2$ when $x = 0$, we have

$$2 = C_1(1) + C_2(0)(1) = C_1.$$

Furthermore, because $y' = 1$ when $x = 0$, we have

$$\begin{aligned} y' &= -2C_1 e^{-2x} + C_2(-2xe^{-2x} + e^{-2x}) \\ 1 &= -2(2)(1) + C_2[-2(0)(1) + 1] \\ 5 &= C_2. \end{aligned}$$

Therefore, the solution is

$$y = 2e^{-2x} + 5xe^{-2x}. \qquad \text{Particular solution}$$

Try checking this solution in the original differential equation.

Higher-Order Linear Differential Equations

For higher-order homogeneous linear differential equations, you can find the general solution in much the same way as you do for second-order equations. That is, you begin by determining the n roots of the characteristic equation. Then, based on these n roots, you form a linearly independent collection of n solutions. The major difference is that with equations of third or higher order, roots of the characteristic equation may occur more than twice. When this happens, the linearly independent solutions are formed by multiplying by increasing powers of x, as demonstrated in Examples 6 and 7.

EXAMPLE 5 Solving a Third-Order Equation

Find the general solution of $y''' - y' = 0$.

Solution The characteristic equation is

$$\begin{aligned} m^3 - m &= 0 \\ m(m-1)(m+1) &= 0 \\ m &= 0, 1, -1. \end{aligned}$$

Because the characteristic equation has three distinct roots, the general solution is

$$y = C_1 + C_2e^{-x} + C_3e^{x}.$$ General solution

EXAMPLE 6 Solving a Third-Order Equation

Find the general solution of $y''' + 3y'' + 3y' + y = 0$.

Solution The characteristic equation is

$$\begin{aligned} m^3 + 3m^2 + 3m + 1 &= 0 \\ (m+1)^3 &= 0 \\ m &= -1. \end{aligned}$$

Because the root $m = -1$ occurs three times, the general solution is

$$y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.$$ General solution

EXAMPLE 7 Solving a Fourth-Order Equation

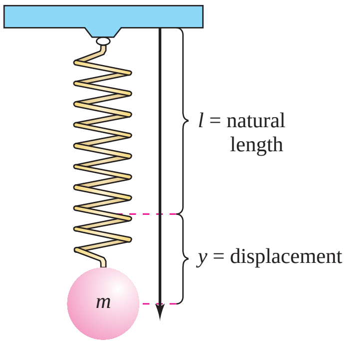
Find the general solution of $y^{(4)} + 2y'' + y = 0$.

Solution The characteristic equation is as follows.

$$\begin{aligned} m^4 + 2m^2 + 1 &= 0 \\ (m^2+1)^2 &= 0 \\ m &= \pm i \end{aligned}$$

Because each of the roots $m_1 = \alpha + \beta i = 0 + i$ and $m_2 = \alpha - \beta i = 0 - i$ occurs twice, the general solution is

$$y = C_1 \cos x + C_2 \sin x + C_3x \cos x + C_4x \sin x.$$ General solution



A rigid object of mass m attached to the end of the spring causes a displacement of y.
Figure 15.9

Applications

One of the many applications of linear differential equations is describing the motion of an oscillating spring. According to Hooke's Law, a spring that is stretched (or compressed) y units from its natural length l tends to *restore* itself to its natural length by a force F that is proportional to y. That is, $F(y) = -ky$, where k is the **spring constant** and indicates the stiffness of the given spring.

Suppose a rigid object of mass m is attached to the end of a spring and causes a displacement, as shown in Figure 15.9. Assume that the mass of the spring is negligible compared with m. If the object is pulled down and released, the resulting oscillations are a product of two opposing forces—the spring force $F(y) = -ky$ and the weight mg of the object. Under such conditions, you can use a differential equation to find the position y of the object as a function of time t. According to Newton's Second Law of Motion, the force acting on the weight is $F = ma$, where $a = d^2y/dt^2$ is the acceleration. Assuming that the motion is **undamped**—that is, there are no other external forces acting on the object—it follows that $m(d^2y/dt^2) = -ky$, and you have

$$\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right)y = 0.$$

Undamped motion of a spring

EXAMPLE 8 Undamped Motion of a Spring

Suppose a 4-pound weight stretches a spring 8 inches from its natural length. The weight is pulled down an additional 6 inches and released with an initial upward velocity of 8 feet per second. Find a formula for the position of the weight as a function of time t.

Solution By Hooke's Law, $4 = k\left(\frac{2}{3}\right)$, so $k = 6$. Moreover, because the weight w is given by mg, it follows that $m = w/g = \frac{4}{32} = \frac{1}{8}$. Hence, the resulting differential equation for this undamped motion is

$$\frac{d^2y}{dt^2} + 48y = 0.$$

Because the characteristic equation $m^2 + 48 = 0$ has complex roots $m = 0 \pm 4\sqrt{3}i$, the general solution is

$$y = C_1e^0 \cos 4\sqrt{3}\,t + C_2e^0 \sin 4\sqrt{3}\,t = C_1 \cos 4\sqrt{3}\,t + C_2 \sin 4\sqrt{3}\,t.$$

Using the initial conditions, you have

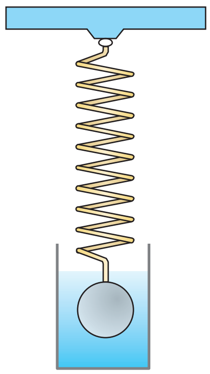
$$\frac{1}{2} = C_1(1) + C_2(0) \quad \Rightarrow \quad C_1 = \frac{1}{2} \qquad y(0) = \tfrac{1}{2}$$

$$y'(t) = -4\sqrt{3}\,C_1 \sin 4\sqrt{3}\,t + 4\sqrt{3}\,C_2 \cos 4\sqrt{3}\,t$$

$$8 = -4\sqrt{3}\left(\frac{1}{2}\right)(0) + 4\sqrt{3}\,C_2(1) \quad \Rightarrow \quad C_2 = \frac{2\sqrt{3}}{3}. \qquad y'(0) = 8$$

Consequently, the position at time t is given by

$$y = \frac{1}{2}\cos 4\sqrt{3}\,t + \frac{2\sqrt{3}}{3}\sin 4\sqrt{3}\,t.$$



A damped vibration could be caused by friction and movement through a liquid.
Figure 15.10

Suppose the object in Figure 15.10 undergoes an additional damping or frictional force that is proportional to its velocity. A case in point would be the damping force resulting from friction and movement through a fluid. Considering this damping force, $-p(dy/dt)$, the differential equation for the oscillation is

$$m\frac{d^2y}{dt^2} = -ky - p\frac{dy}{dt}$$

or, in standard linear form,

$$\frac{d^2y}{dt^2} + \frac{p}{m}\left(\frac{dy}{dt}\right) + \frac{k}{m}y = 0.$$

Damped motion of a spring

EXERCISES FOR SECTION 15.3

In Exercises 1–4, verify the solution of the differential equation.

	Solution	*Differential Equation*
1.	$y = (C_1 + C_2x)e^{-3x}$	$y'' + 6y' + 9y = 0$
2.	$y = C_1e^{2x} + C_2e^{-2x}$	$y'' - 4y = 0$
3.	$y = C_1\cos 2x + C_2\sin 2x$	$y'' + 4y = 0$
4.	$y = e^{-x}\sin 3x$	$y'' + 2y' + 10y = 0$

In Exercises 5–30, find the general solution of the linear differential equation.

5. $y'' - y' = 0$
6. $y'' + 2y' = 0$
7. $y'' - y' - 6y = 0$
8. $y'' + 6y' + 5y = 0$
9. $2y'' + 3y' - 2y = 0$
10. $16y'' - 16y' + 3y = 0$
11. $y'' + 6y' + 9y = 0$
12. $y'' - 10y' + 25y = 0$
13. $16y'' - 8y' + y = 0$
14. $9y'' - 12y' + 4y = 0$
15. $y'' + y = 0$
16. $y'' + 4y = 0$
17. $y'' - 9y = 0$
18. $y'' - 2y = 0$
19. $y'' - 2y' + 4y = 0$
20. $y'' - 4y' + 21y = 0$
21. $y'' - 3y' + y = 0$
22. $3y'' + 4y' - y = 0$
23. $9y'' - 12y' + 11y = 0$
24. $2y'' - 6y' + 7y = 0$
25. $y^{(4)} - y = 0$
26. $y^{(4)} - y'' = 0$
27. $y''' - 6y'' + 11y' - 6y = 0$
28. $y''' - y'' - y' + y = 0$
29. $y''' - 3y'' + 7y' - 5y = 0$
30. $y''' - 3y'' + 3y' - y = 0$

31. Consider the differential equation $y'' + 100y = 0$ and the solution $y = C_1\cos 10x + C_2\sin 10x$. Find the particular solution satisfying each of the following initial conditions.
(a) $y(0) = 2,\ y'(0) = 0$
(b) $y(0) = 0,\ y'(0) = 2$
(c) $y(0) = -1,\ y'(0) = 3$

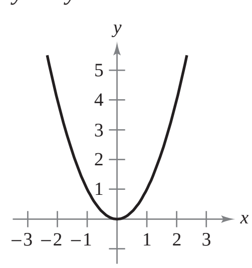
32. Determine C and ω such that $y = C\sin\sqrt{3}\,t$ is a particular solution of the differential equation $y'' + \omega y = 0$, where $y'(0) = -5$.

In Exercises 33–36, find the particular solution of the linear differential equation.

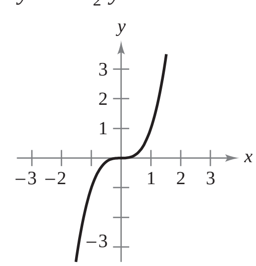
33. $y'' - y' - 30y = 0$
$y(0) = 1,\ y'(0) = -4$
34. $y'' + 2y' + 3y = 0$
$y(0) = 2,\ y'(0) = 1$
35. $y'' + 16y = 0$
$y(0) = 0,\ y'(0) = 2$
36. $y'' + 2y' + 3y = 0$
$y(0) = 2,\ y'(0) = 1$

Think About It **In Exercises 37 and 38, give a geometric argument to explain why the graph cannot be a solution of the differential equation. It is not necessary to solve the differential equation.**

37. $y'' = y'$



38. $y'' = -\frac{1}{2}y'$



Vibrating Spring **In Exercises 39–44, describe the motion of a 32-pound weight suspended on a spring. Assume that the weight stretches the spring $\frac{2}{3}$ foot from its natural position.**

39. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and released.

40. The weight is raised $\frac{2}{3}$ foot above the equilibrium position and released.

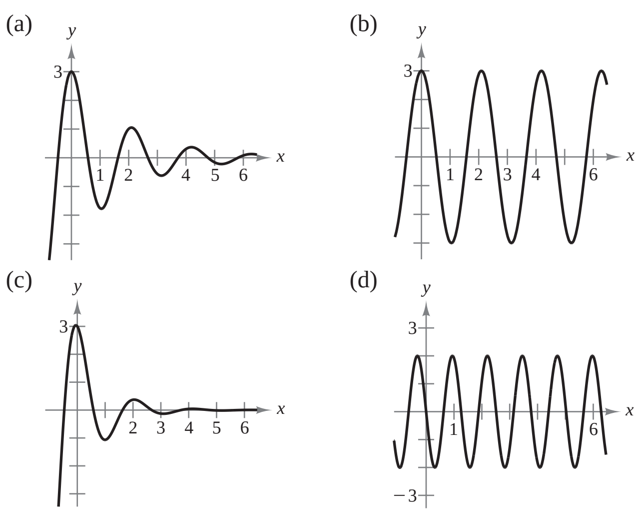
41. The weight is raised $\frac{2}{3}$ foot above the equilibrium position and started off with a downward velocity of $\frac{1}{2}$ foot per second.

42. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and started off with an upward velocity of $\frac{1}{2}$ foot per second.

43. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{8}$ speed at all times.

44. The weight is pulled $\frac{1}{2}$ foot below the equilibrium position and released. The motion takes place in a medium that furnishes a damping force of magnitude $\frac{1}{4}|v|$ at all times.

Vibrating Spring **In Exercises 45–48, match the differential equation with the graph of a particular solution. [The graphs are labeled (a), (b), (c), and (d).] The correct match can be made by comparing the frequency of the oscillations or the rate at which the oscillations are being damped with the appropriate coefficient in the differential equation.**



45. $y'' + 9y = 0$

46. $y'' + 25y = 0$

47. $y'' + 2y' + 10y = 0$

48. $y'' + y' + \frac{37}{4}y = 0$

49. If the characteristic equation of the differential equation

$$y'' + ay' + by = 0$$

has two equal real roots given by $m = r$, show that

$$y = C_1e^{rx} + C_2xe^{rx}$$

is a solution.

50. If the characteristic equation of the differential equation

$$y'' + ay' + by = 0$$

has complex roots given by $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$, show that

$$y = C_1e^{\alpha x}\cos\beta x + C_2e^{\alpha x}\sin\beta x$$

is a solution.

True or False? **In Exercises 51–54, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

51. $y = C_1e^{3x} + C_2e^{-3x}$ is the general solution of $y'' - 6y' + 9 = 0$.

52. $y = (C_1 + C_2x)\sin x + (C_3 + C_4x)\cos x$ is the general solution of $y^{(4)} + 2y'' + y = 0$.

53. $y = x$ is a solution of $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdot\cdot\cdot + a_1y' + a_0y = 0$ if and only if $a_1 = a_0 = 0$.

54. It is possible to choose a and b such that $y = x^2e^x$ is a solution of $y'' + ay' + by = 0$.

The *Wronskian* of two differentiable functions f and g, denoted by $W(f, g)$, is defined as the function given by the determinant

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

The functions f and g are linearly independent if there exists at least one value of x for which $W(f, g) \neq 0$. In Exercises 55–58, use the Wronskian to verify the linear independence of the two functions.

55. $y_1 = e^{ax}$
$y_2 = e^{bx},\ a \neq b$

56. $y_1 = e^{ax}$
$y_2 = xe^{ax}$

57. $y_1 = e^{ax}\sin bx$
$y_2 = e^{ax}\cos bx,\ b \neq 0$

58. $y_1 = x$
$y_2 = x^2$

59. Euler's differential equation is of the form

$$x^2y'' + axy' + by = 0, \quad x > 0$$

where a and b are constants.

(a) Show that this equation can be transformed into a second-order linear equation with constant coefficients by using the substitution $x = e^t$.

(b) Solve $x^2y'' + 6xy' + 6y = 0$.

60. Solve

$$y'' + Ay = 0$$

where A is constant, subject to the conditions $y(0) = 0$ and $y(\pi) = 0$.