



Constructions

The Greeks were fascinated by the challenge of what we call construction problems. **Constructions** involve making geometric figures using only a compass and a straightedge—in other words, without a protractor or ruler.

Consider these three famous constructions, which were first attempted over 2,000 years ago:

1. Construct an angle whose measure is one-third the measure of a given angle; this is often called the trisect.
2. Construct a square whose area is equal to the area of a given circle.
3. Given the face of a cube, construct the face of a cube whose volume is exactly double the volume of the first cube.

Some of the greatest mathematicians in the world grappled with these constructions. It was not until the early 1800s that the brilliant French mathematician Evariste Galois proved that they are indeed impossible.


To the modern student, these problems may seem silly or pointless. However, the pursuit of some of the more difficult constructions is directly responsible for some major discoveries in mathematics, including the theory of equations and group theory.¹

A reasonable question that many of my students ask is, so why do constructions now? There are several reasons. First, there is rich opportunity for problem-solving, reasoning, and making connections. Another is that many students who struggle with computational mathematics love constructions. A third is that it can be fun, and some very pretty figures can be constructed.

In this appendix, we will cover some basic constructions. The exercises offer some interesting challenges that call upon you to apply your reasoning and problem-solving tools.

¹Vivian Shaw Groza, *A Survey of Mathematics: Elementary Concepts and Their Historical Development* (New York: Holt, Rinehart and Winston, 1968), p. 149.

Construction 1 Copying a Line Segment

The first construction we will consider is simply copying a line segment. We need this when constructing congruent triangles and other polygons. If you have a given line segment, how could you make another line segment that was the same length using only a compass and a straightedge? Think before reading on. . . . 

This is a fairly straightforward construction.

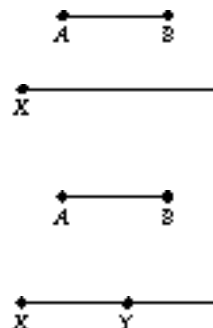
Let \overline{AB} be the given line segment.

Begin by drawing another segment, and call one endpoint X .

Open the compass so that the point lies on A and the pencil point¹ lies on B .

Without changing the opening of the compass, place the point of the compass on X and make a mark with the pencil of the compass. Call that point Y .

$$\overline{AB} \cong \overline{XY}$$



Construction 2 Copying a Triangle when Three Sides Are Given

We can apply this construction to **copy any triangle**. From the definition of congruent triangles, in order to know for sure that two triangles are congruent, we need to verify that all six pairs of corresponding parts— three pairs of sides and three pairs of angles— are congruent. Do we need all six? Let us see what the following construction tells us.

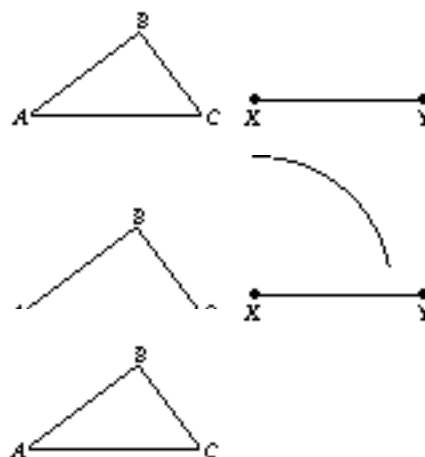
Let ABC be the given triangle.

Copy line segment \overline{AC} and label it \overline{XY} .

Open the compass so that the point lies on A and the pencil point lies on B .

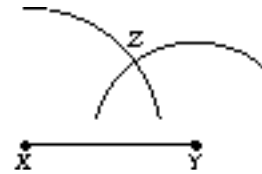
With the same opening and the point on X , draw an arc.

Open the compass so that the point lies on C and the pencil point lies on B .



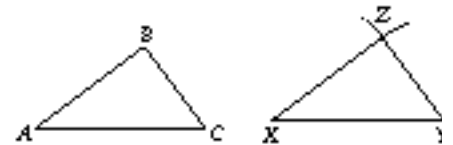
¹Language note: We will refer to the point of the compass that writes as the *pencil point* and the other point simply as *the point*.

With the same opening and the point on Y , draw an arc. Call the point at which this arc intersects the other arc point Z .



Connect points X and Z . Connect points Y and Z .

$$\triangle ABC \cong \triangle XYZ$$



Thus, we made a copy of $\triangle ABC$ by making $\overline{XY} \cong \overline{AC}$, $\overline{ZY} \cong \overline{BC}$, and $\overline{XZ} \cong \overline{AB}$.

It turns out that if the three sides are congruent, the two triangles must be congruent. This is known in geometry as the side-side-side congruence property (SSS).

Thus, we have discovered that we can make triangles without having to know that all six pairs of corresponding parts are congruent!

Construction 3 Copying an Angle

Sometimes we need to copy an angle. How might we do that? What do you think?

Let angle A be the given angle.

Begin by drawing a line segment and labeling one endpoint O .

With the point of the compass on point A , draw an arc that intersects each side of the angle. Call those points B and C .

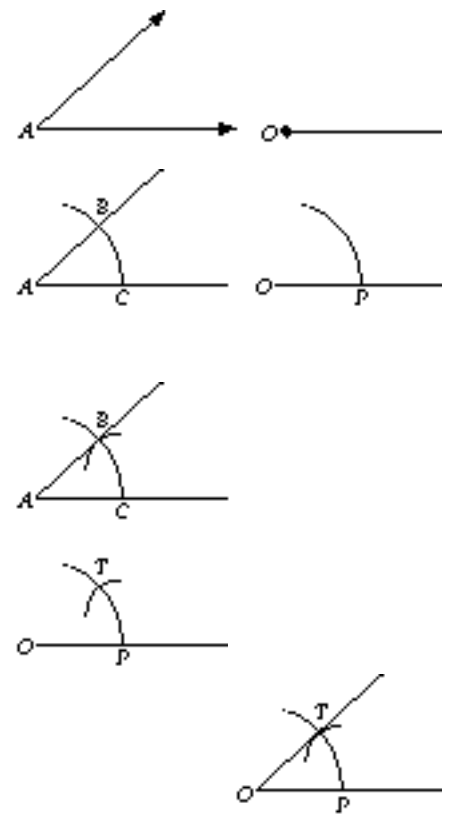
Now place the point of the compass on point O and draw an arc with the same radius so that it intersects the line segment. Call that intersection point P .

Open the compass so that the point lies on C and the pencil point lies on B .

Keeping the compass at the same setting place the point of the compass on point P and draw an arc that intersects the arc you drew before. Call the point of intersection of the two arcs T .


Connect points O and T .

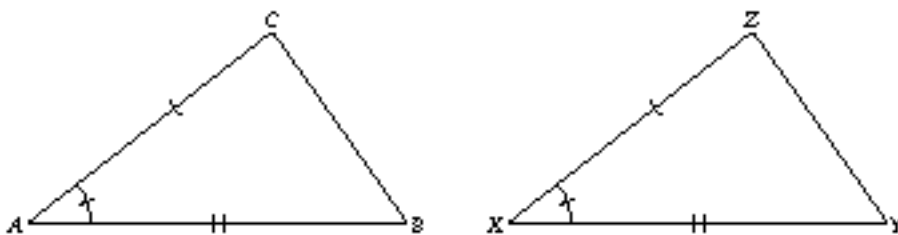
$$\sphericalangle BAC \cong \sphericalangle TOP$$



Construction 4 Copying a Triangle when Two Sides and the Angle Between Are Given

We just found that if all three pairs of corresponding sides are congruent, two triangles must also be congruent. What if we knew that two pairs of corresponding sides were congruent and that the angles between these pairs of sides were congruent? Must the two triangles be congruent? If so, this would be known as the SAS property. That is, if you copy two sides and the angle between them, *must* the two triangles be congruent?

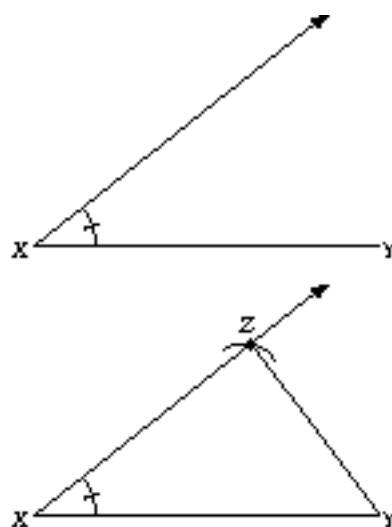
For example, suppose $\triangle ABC$ is the given triangle. If we construct $\triangle XYZ$ so that $\overline{AB} \cong \overline{XY}$, $\overline{AC} \cong \overline{XZ}$, and $\sphericalangle A \cong \sphericalangle X$, must the copied triangle be congruent to $\triangle ABC$? What do you think? Try this yourself before reading on. . . 



Note: The marks on the triangles are visual aids to represent the congruent pairs. That is, sides \overline{AC} and \overline{XZ} have one mark, sides \overline{AB} and \overline{XY} have two marks, and angles A and X each have one mark.

First, we copy \overline{AB} so that $\overline{XY} \cong \overline{AB}$.
 Then we copy angle A so that $\sphericalangle X \cong \sphericalangle A$.
 Next we copy \overline{AC} so that $\overline{XZ} \cong \overline{AC}$.


The only thing remaining is to connect Y and Z.

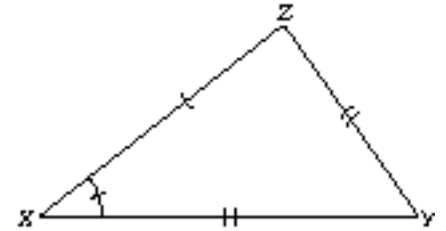
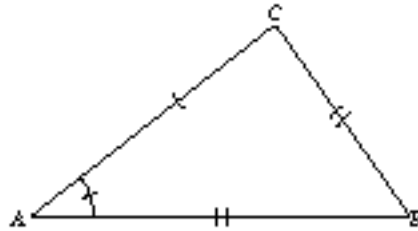


It turns out that if we copy two sides of a triangle and the angle between them, the new triangle must be congruent to the first triangle. Therefore, the side-angle-side congruence property (SAS), proposed above, holds true.

Construction 5 Copying a Triangle when Two Sides and an Angle That Is Not Between Those Sides Are Given

What if we knew that two pairs of corresponding sides were congruent and that an angle that was not between these pairs of sides was congruent? Must

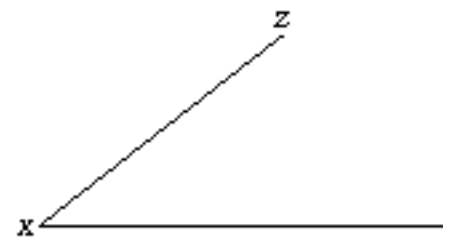
the two triangles be congruent? If so, this would be known as the SSA property. That is, if you copy two sides and an angle not between them, *must* the two triangles be congruent? For example, suppose $\triangle ABC$ is the given triangle. What if we copy $\angle A$, \overline{AC} , and \overline{CB} ? What do you think? Try this yourself before reading on. . . . 



We can begin by copying $\angle A$ so that $\angle A \cong \angle X$.

Then we can copy \overline{AC} so that $\overline{XZ} \cong \overline{AC}$.

Finally, we can copy \overline{BC} so that $\overline{YZ} \cong \overline{BC}$.



However, with the point of the compass on Z , we find that the arc crosses \overline{XY} at two points.

What does this mean?

It means, unfortunately, that the SSA property does not hold water. Why is this?

If we compare $\triangle ABC$ to $\triangle XY_2Z$,

$$\overline{AC} \cong \overline{XZ}$$

$$\overline{BC} \cong \overline{Y_2Z}$$

$$\text{and } \angle A \cong \angle Z$$

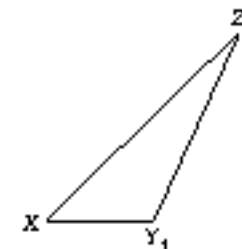
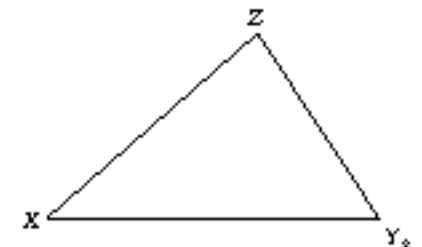
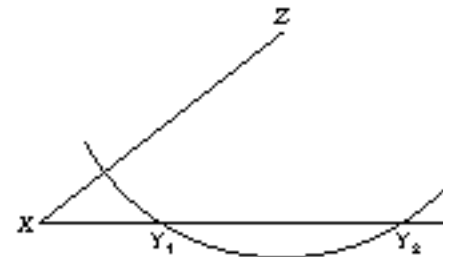
If we compare $\triangle ABC$ to $\triangle XY_1Z$,

$$\overline{AC} \cong \overline{XZ}$$

$$\overline{BC} \cong \overline{Y_1Z}$$

$$\text{and } \angle A \cong \angle Z$$

That is, we have constructed two different triangles from two sides and a nonincluded angle of the given triangle.



Thus, we have seen that there is not an SSA triangle congruence property. What about ASA? What about AAS? These will be left as exercises.

In order to make some of the more elaborate designs found in the exercises, we need several more constructions.

Construction 6 Bisecting a Given Line Segment

Some constructions require us to bisect a line segment. How might we do that?

Let \overline{TP} be the given segment.

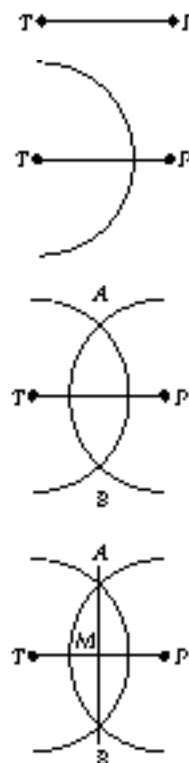
With the point of the compass on T , make an arc above and below the line segment. *Note:* The length of the compass opening must be greater than half the length of the line segment. Why is this?

Now place the point of the compass on P and make an arc above and below the line segment.

If you made the arcs large enough, they will intersect at two points. Call those points A and B .

Place the straightedge so that it contains points A and B . The point at which the straightedge intersects \overline{TP} will be the midpoint of \overline{TP} . That point is called M in the diagram.

Furthermore, \overleftrightarrow{AB} is perpendicular to \overleftrightarrow{TP} . Do you see why?



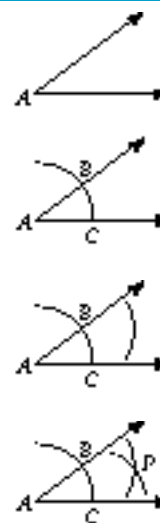
Construction 7 Bisecting an Angle

Sometimes we need to be able to bisect an angle. Let A be the given angle.

With the point of the compass on point A , draw an arc that intersects each side of the angle. Call those points B and C .

With the point of the compass on point B , draw an arc.

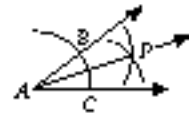
With the point of the compass on point C , draw an arc with the same radius. Call the point at which the two arcs intersect point P .




Draw ray \overrightarrow{AP} .

Angle A has now been bisected.
Thus,

$$m\angle BAP = m\angle PAC$$



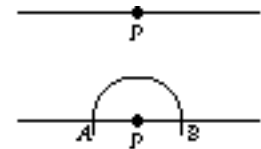
Can you prove that the angle has been bisected—that is, that $m\angle BAP = m\angle PAC$? Think about it before reading on. . . 

$\overline{AB} \cong \overline{AC}$	Constructed that way
$\overline{BP} \cong \overline{CP}$	Constructed that way
$\overline{AP} \cong \overline{AP}$	Any segment is congruent to itself.
$\triangle ABP \cong \triangle ACP$	SSS
$m\angle BAP = m\angle PAC$	They are corresponding angles of congruent triangles.

Construction 8 Constructing a Perpendicular to a Given Line Segment Through a Given Point

Sometimes we have a line segment and we want to make a right angle—for example, when constructing a right triangle or a square. In that case, we want to construct a line segment perpendicular to a given line segment through a given point.

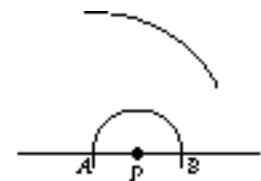
Begin with a line segment and a point P that lies on the line segment.



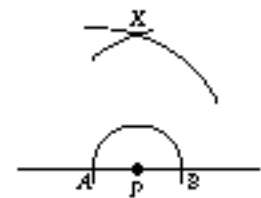
Place the point of the compass on point P and draw an arc so that it intersects the line segment on either side of P .

Call those points of intersection A and B .

Open the compass wider and with the point of the compass on point A , draw an arc above the line segment.

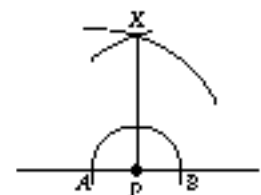


Place the point of the compass on point B , keeping the same setting, and draw an arc above the line segment. Call the point of intersection X .



Connect points X and P .

$$\overline{XP} \perp \overline{AB}$$



Construction 9 Constructing a Perpendicular from a Point to a Line

In some cases, we have a point and we want to connect that point to a line segment so that perpendicular lines are formed. In that case, we want to construct a perpendicular from a point to a line.

Begin with a line and a point not on the line.

Open the compass so that its radius is great enough so that with one point of the compass on P , an arc will intersect the line segment at two points, which we will call A and B . Draw arc AB .

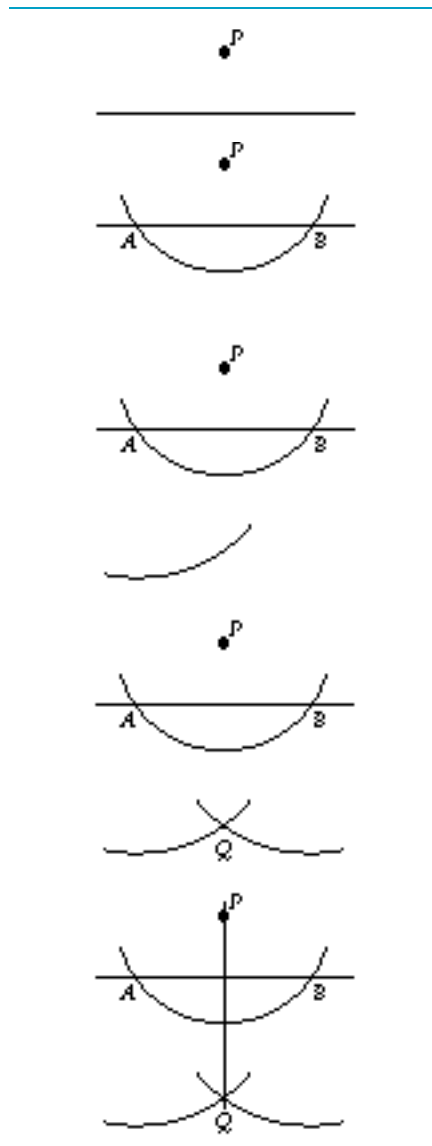
Now open the compass so that its radius is appreciably greater than half the distance from A to B .

Place the point of the compass on point A and draw an arc below the line segment.

Place the point of the compass on point B and draw an arc below the line segment. If the previous arc was long enough, the two arcs will intersect. Call the intersection point Q .

Connect points P and Q .

$$\overline{AB} \perp \overline{PQ}$$



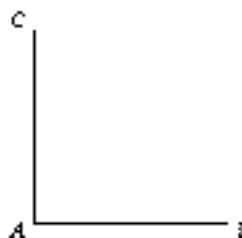
Do you see connections between the previous two constructions and Construction 6?

Construction 10 Constructing a Square

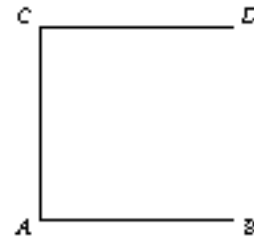
Earlier we discovered several postulates that enable us to construct triangles or to copy a given triangle. We now have enough tools to construct a square.

There are many ways to construct a square.

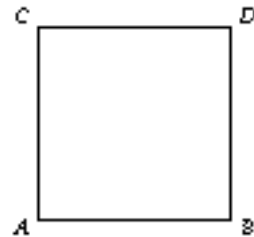
For example, we could start with line segment AB . We could construct AC perpendicular to AB and having the same length.



Then we could construct \overline{CD} perpendicular to \overline{AC} and having the same length.



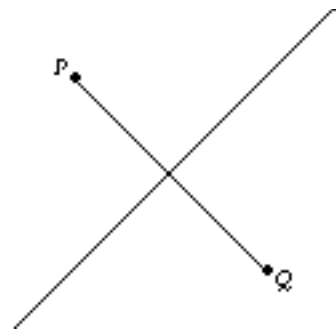
Then we could connect D and B .



However, there are many other ways to construct a square—easier ones, too. Try to develop a way to construct a square yourself before reading on. . . . 

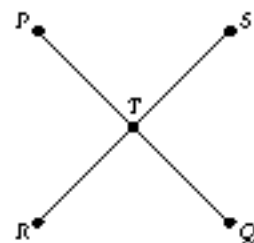
ANOTHER WAY TO CONSTRUCT A SQUARE

One construction would be to start with a line segment \overline{PQ} and construct a perpendicular bisector as we did in Construction 9. Call the intersection point T .



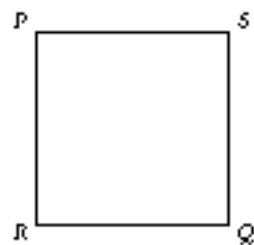
Then open the compass to the desired length, that is, the length of \overline{PT} .

Mark points R and S so that \overline{PT} , \overline{ST} , \overline{QT} , and \overline{RT} are all congruent.



Connect points $PSQR$, and voilà, you have a square.

Do you see why this is a square?



How many other ways can you find to construct a square?
 Johann says this isn't quite "legal." He says that this technique won't work if you have to construct a square congruent to a given square, let's say square $ABCD$ above. What do you think? Will this technique work? If it does not work directly, can it be adapted, or is Johann right?

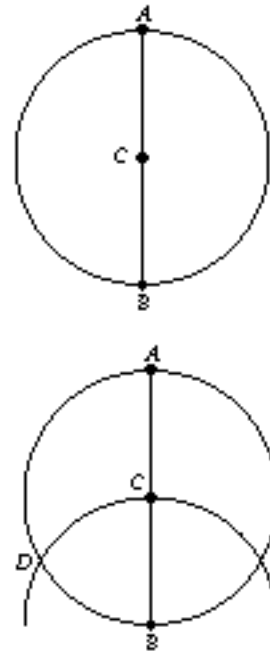
Construction 11 Inscribing an Equilateral Triangle

Many of the elaborate constructions found in the exercises begin by inscribing a figure in a circle. The three constructions below enable us to inscribe an equilateral triangle in a circle, a square in a circle, and a hexagon in a circle.

Draw a circle and call the center C .
 Make a diameter and call the two endpoints A and B .

Open the compass so that its opening is equal to the radius of the circle.
 Place the point of the compass on point B and draw an arc that intersects the circle in two places. Call those points D and E .

If you now connect points A , D , and E , you have an equilateral triangle.
 The proof that this indeed is an equilateral triangle will be left as an exercise.



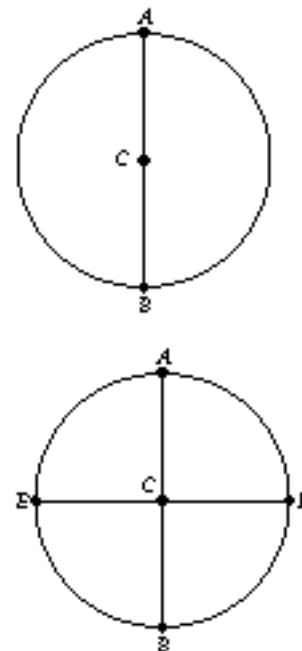
Construction 12 Inscribing a Square

Draw a circle and call the center C .
 Make a diameter and call the two endpoints A and B .

Construct another diameter that is perpendicular to \overline{AB} . (That is, construct the perpendicular bisector of \overline{AB} .) Call this diameter \overline{EF} .

Connect the points to make square $AEBF$.

The proof that this indeed is a square will be left as an exercise.



Construction 13 Inscribing a Hexagon

Draw a circle and call the center C . Make a diameter and call the two endpoints A and B .

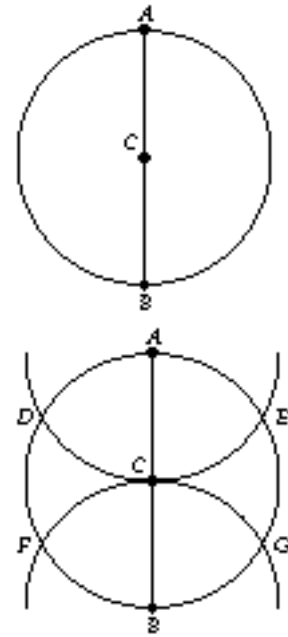
Open the compass so that its opening is equal to the radius of the circle.

Place the point of the compass on point A and draw an arc that intersects the circle in two places. Call those points D and E .

Place one point of the compass on point B and draw an arc that intersects the circle in two places. Call those points F and G .


Connect the points to make hexagon $ADFBGE$.

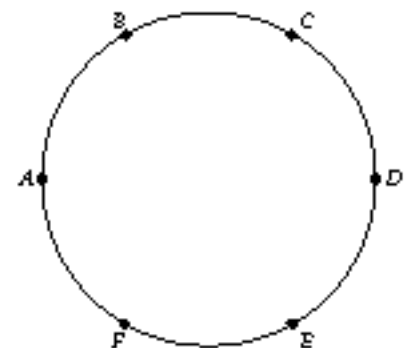
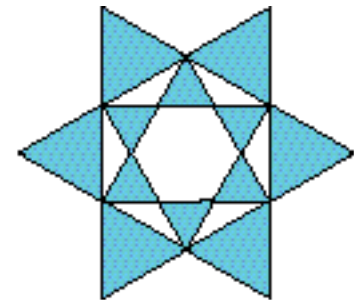
The proof that this is a regular hexagon will be left as an exercise.



Construction 14 Construct This Figure

Now we have the tools to construct the following figure, the figures in the exercises, and a whole world of other figures that you can create. This construction not only serves to continue to develop your problem-solving and reasoning skills, but will also serve as an assessment of your understanding of the constructions we have developed. If you own the constructions that we have studied so far, you will be better able to apply that knowledge to new problems that are not just like the ones we have done before.

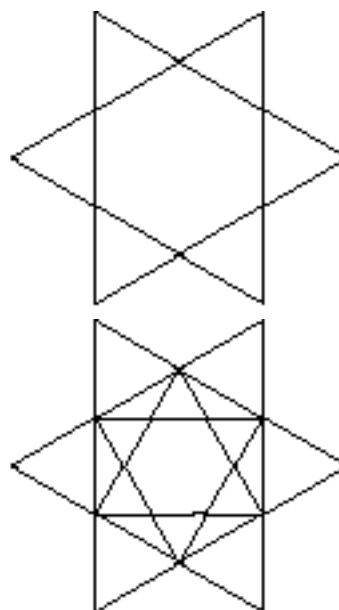
Can you figure out how to make this figure using only your compass and straightedge? Try to do so before reading on. . . 



Construct a regular hexagon, but do not connect the points to make a hexagon.

Instead, connect points BDF and points ACE to make two equilateral triangles.

Within the hexagon in the middle of the figure, connect points so as to make two equilateral triangles.



If you now shade in the appropriate regions, you will have the desired figure.

EXERCISES

- There are three more congruence properties for triangles: ASA, AAS (that is, two pairs of angles are congruent and one pair of sides not between the two angles is congruent), and HL (that is, with right triangles, the two hypotenuses are congruent and one pair of legs is congruent).
 - Justify the ASA property for triangles.
 - Justify the AAS property for triangles.
 - Justify the HL property for triangles.
- Justify the following constructions:
 - Construction of a perpendicular bisector
 - Construction of a perpendicular from a point not on a line
 - Inscribing an equilateral triangle in a circle
 - Inscribing a square in a circle
- In Chapter 8, an isosceles triangle was defined as a triangle having at least two congruent sides. It was also mentioned that it can be proved that the base angles of isosceles triangles are equal. Prove that now. *Note:* There is more than one possible way to do so.
- Draw a scalene triangle. Construct the midpoints and connect them to form a triangle. Is there a relationship between this new triangle and the original one?
 - Construct an isosceles triangle. Construct the midpoints and connect them to form a triangle. Is there a relationship between this new triangle and the original one?
 - Construct an equilateral triangle. Construct the midpoints and connect them to form a triangle. Is there a relationship between this new triangle and the original one?
- Construct a right triangle. Construct the midpoints and connect them to form a triangle. Is there a relationship between this new triangle and the original one?
- Do you think there is an SSSS congruence property for quadrilaterals? Write your initial response and justification.
 - Design and test a way to prove or refute this hypothesis. Justify your assertion.
- Draw a variety of different quadrilaterals. In each quadrilateral, construct the midpoints of the two sides and then connect the midpoints to create a new quadrilateral inside the original one.

What generalizations and conclusions can you make about the quadrilateral formed by connecting the midpoints of various quadrilaterals? For example, if the given quadrilateral is convex, must the new one also be convex? If the given quadrilateral is a rectangle, must the new quadrilateral be a rectangle?
- Using your ruler, first draw line segments of the following lengths: 3 inches, 4 inches, 5 inches, 6 inches, and 8 inches. Then use the compass and straightedge to construct the following triangles.
 - 3 inches, 4 inches, and 5 inches
 - 3 inches, 4 inches, and 6 inches
 - 3 inches, 4 inches, and 8 inches
 - Describe your conclusions from these constructions.
- There are many applications of congruent triangles in real life. For example, we can use congruent triangles to

determine the distance across a lake. Explain and justify one such method. *Note:* There is more than one possible way to do so.

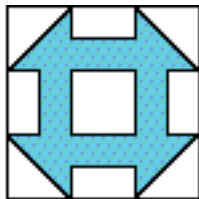


9. Can you construct the following figures? If you can, do it and explain how you did it. If you can't, explain why it is not possible.

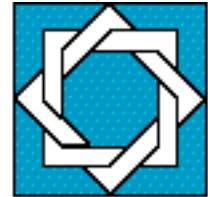
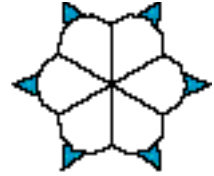
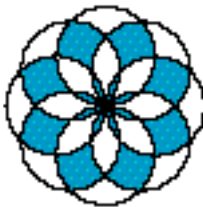
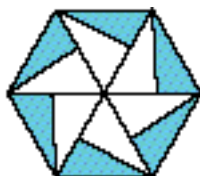
a.



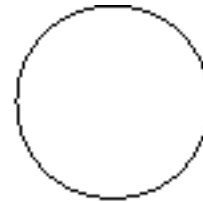
b.



10. Make the following designs.² After you have figured out how to do so, write the directions for doing so. (Before turning in your directions, you might want to give them to a fellow student and see if that student can follow your directions.) Justify the construction.



11. How might you find the center of the circle below? You are encouraged to try constructions, but you are not limited to constructions. How many different ways can you find? Justify each way.



You may want to trace or construct circles on a sheet of paper and experiment.

12. In this section, we learned how to inscribe geometric figures within a circle. What if we began with the figure and circumscribed a circle around the figure?

- a. Determine a method for circumscribing a circle around an equilateral triangle.
- b. Determine a method for circumscribing a circle around any triangle or explain why you think it is not possible.
- c. Determine a method for circumscribing a circle around a square.
- d. Determine a method for circumscribing a circle around a rectangle.
- e. Determine a method for circumscribing a circle around a parallelogram.

²Dale Seymour and Reuben Schadler, *Creative Constructions* (Palo Alto, Calif.: Creative Publications, 1974).

